CATEGORY THEORY
TOPIC 34: TOPOLOGICAL SPACES
(DRAFT)

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ABSTRACT. We define *topological spaces*, which are sets together with the additional structure provided by a collection of open sets, give some basic examples, and develop their initial properties.

1. Topological Spaces

We have seen that the concept of *neighborhood* in a metric space may be used to define basic notions such as interior, boundary, closure, and accumulation points, as well as to define and generalize the notions of open sets and continuous functions. We have shown that the collection of open sets is closed under arbitrary unions and finite intersections. It turns out that these properties alone are sufficient to develop a theory that is powerful enough to replace much of the computation involved in analytical proofs, through the theory of topology.

Definition 1. A *topological space* is a set X together with a collection of subsets $\mathcal{T} \subset \mathcal{P}(X)$ such that

(T1) $\varnothing \in \mathcal{T}$ and $X \in \mathcal{T}$;
(T2) $\mathcal{U} \subset \mathcal{T} \Rightarrow \cup\mathcal{U} \in \mathcal{T}$;
(T3) $\mathcal{U} \subset \mathcal{T}$ and $\mathcal{U}$ finite $\Rightarrow \cap\mathcal{U} \in \mathcal{T}$.

The collection $\mathcal{T}$ is called a *topology* on X.

A subset $A \subset X$ is called *open* if $A \in \mathcal{T}$, and is called *closed* if $X \smallsetminus A \in \mathcal{T}$.

Example 1. Let X be a set and let $\mathcal{T} = \{\varnothing, X\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *trivial* topology on X.

Example 2. Let X be a set and let $\mathcal{T} = \mathcal{P}(X)$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *discrete* topology on X.

Example 3. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is finite }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cofinite* topology on X.

Example 4. Let X be a set and let $\mathcal{T} = \{A \subset X \mid X \smallsetminus A \text{ is countable }\}$. Then $(X, \mathcal{T})$ is a topological space and $\mathcal{T}$ is called the *cocountable* topology on X.

Definition 2. Let X be a set. A *tower* of subsets of X is a collection $\mathcal{T} \subset \mathcal{P}(X)$ which contains the empty set and the entire set and is totally ordered by inclusion.

Example 5. Let X be a set and $\mathcal{T}$ a tower of subsets of X. Then $\mathcal{T}$ is a topology on X, called a *tower topology.*

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Example 6. Let X be a totally ordered set. For $a \in X$, set

$$L_a = \{x \in X \mid x < a\} \quad \text{and} \quad R_a = \{x \in X \mid x > a\}.$$

Set

$$\mathcal{L} = \{L_a \mid a \in X\} \cup \{\varnothing, X\} \quad \text{and} \quad \mathcal{R} = \{R_a \mid a \in X\} \cup \{\varnothing, X\}.$$

Then $\mathcal{L}$ is a topology on X, called the *left order topology*, and $\mathcal{R}$ is a topology on X, called the *right order topology*.

Example 7. Let (X, ρ) be a metric space. Let $U \subset X$ and say that U is *open* if for every $u \in U$ there exists $\epsilon > 0$ such that $x \in U$ whenever $\rho(x, u) < \epsilon$. Let $\mathcal{T}$ denote the collection of open sets. Then $(X, \mathcal{T})$ is a topological space.

2. Neighborhoods

Definition 3. Let X be a topological space and let $x \in X$. A *neighborhood* of x is a subset $N \subset X$ such that there exists an open set $U \subset N$ with $x \in U$.

Remark 1. Let X be a topological space and let $x \in X$. It is immediate that if N is a neighborhood of x, then $x \in N$.

If U is an open set containing x, then U is itself a neighborhood of x, and is referred to as an open neighborhood. Thus there exists at least one neighborhood of x; indeed, X is open and contains x.

We are interested in sets A whose intersection with neighborhoods of x are nonempty, in which case we say that A intersects the neighborhood. If $x \in A$, then every neighborhood of x intersects A; this is the less interesting case for us.

Clearly A intersects every neighborhood of x if and only if A intersects every open neighborhood of x; the forward direction is immediate and the reverse direction is given by considering a neighborhood which does not intersect A, which must contain an open neighborhood which does not intersect A.

Definition 4. A *deleted neighborhood* of x is a set of the form $N \smallsetminus \{x\}$, where N is a neighborhood of x.

Remark 2. Let X be a topological space and let $x \in X$. If $N \smallsetminus \{x\}$ is a deleted neighborhood of x which does not intersect A, then either N does not intersect A or there is an open set $U \subset N$ such that x is the only element of A in that open set.

3. Classification of Points

3.1. Interior Points.

Definition 5. Let X be a topological space and let $A \subset X$. An *interior point* of A is a point $x \in X$ such that A contains a neighborhood of x. The *interior* of A is the set of interior points of A and is denoted A°.

Proposition 1. *Let X be a topological space and let $A \subset X$. Then $A^\circ \subset A$.*

Proof. Let $x \in A^\circ$. Then there exists a neighborhood of x which is contained in A. Since x is in this neighborhood, it is the case that $x \in A$. So $A^\circ \subset A$. □

Proposition 2. *Let X be a topological space and let $A \subset X$. Then A° open.*

Proof. Let $a \in A^\circ$. Then a is an interior point of A, so there exists an open neighborhood U_a of a such that $U_a \subset A$. If $u \in U_a$, then U_a is also an open neighborhood of u, so that $u \in A^\circ$; thus $U_a \subset A^\circ$.

Thus for each $a \in A^\circ$, let U_a be a open neighborhood of a which is contained in A°. Let $U = \bigcup_{a \in A} U_a$; since U is a union of open sets, U is open.

We claim that $A^\circ = U$. To see this, consider that if $a \in A^\circ$, then $a \int U_a$, so $a \in U$, so $A^\circ \subset U$. On the other and, if $u \in U$, then $u \in U_a$ for some a, and $U_a \subset A^\circ$, so $u \in A^\circ$, so $U \subset A^\circ$. Thus $U = A^\circ$, and since U is a union of open sets, A° is open. □

Proposition 3. *Let X be a topological space and let $A \subset X$. Then A is open if and only if $A = A^\circ$.*

Proof. Suppose A is open. We already know that $A^\circ \subset A$. Since A is a neighborhood of every point in A, every point in A is an interior point, so $A \subset A^\circ$.

Suppose that $A = A^\circ$. Since A° is open, so is A. □

Proposition 4. *Let X be a topological space and let $A \subset X$. Then A is open if and only if every point in A is an interior point.*

Proof. This is just a rewording of the previous proposition. □

Proposition 5. *Let X be a topological space and let $A \subset X$. The interior of A is the union of all open sets which are contained in A.*

Proof. Let $\mathcal{U} = \{U \subset X \mid U \text{ is open and } U \subset A\}$. We wish to show that $A^\circ = \cup\mathcal{U}$.

Since A° is an open set which is contained in A, $A^\circ \in \mathcal{U}$, so $A^\circ \subset \cup\mathcal{U}$.

On the other hand, every point $u \in \mathcal{U}$ is in U for some $U \in \mathcal{U}$, so U is a neighborhood of u which is contained in A, so u is an interior point of A, ad $u \in A^\circ$; thus $\cup\mathcal{U} \subset A^\circ$. □

Proposition 6. *Let X be a topological space and let $A \subset X$. Then*

- **(a)** $A \subset B \Rightarrow A^\circ \subset B^\circ$;
- **(b)** $(A^\circ)^\circ = A^\circ$;
- **(c)** $(A \cap B)^\circ = A^\circ \cap B^\circ$.

Proof. Exercise. □

3.2. **Closure Points.**

Definition 6. Let X be a topological space and let $A \subset X$. A *closure point* of A is a point $x \in X$ such that every neighborhood of x intersects A. The *closure* of $A \subset X$ is the set of closure points of A and is denoted $\overline{A}$.

Proposition 7. *Let X be a topological space and $A \subset X$. Then $A \subset \overline{A}$.*

Proof. Let $a \in A$. Every neighborhood of a contains a, and since $a \in A$, every neighborhood of a intersects A. Thus a is a point of closure of A, so $a \in \overline{A}$. □

Proposition 8. *Let X be a topological space and $A \subset X$. Then $\overline{A}$ is closed.*

Proof. Wish to show that $X \smallsetminus \overline{A}$ is open. Thus let $u \in X \smallsetminus \overline{A}$, so that u is not a closure point of A. This means that there exists a neighborhood U of u which does not intersect A. If $v \in U$, then U is a neighborhood of v which does not intersect A, so v is not a closure point of A. Hence $U \subset X \smallsetminus \overline{A}$, which shows that u is an interior point of $X \smallsetminus \overline{A}$, so $X \smallsetminus \overline{A}$ is open. Therefore $\overline{A}$ is closed. □

Proposition 9. *Let X be a topological space and $A \subset X$. Then A is closed if and only if $A = \overline{A}$.*

Proof. Suppose A is closed. We have already seen that $A \subset \overline{A}$. Now suppose that $u \notin A$. Then $u \in X \smallsetminus A$, which is open, so there exists a neighborhood U of u which is contained in $X \smallsetminus A$, so U does not intersect A. Thus u is not a closure point of A, so $u \notin \overline{A}$. Thus $a \in A$ if and only if $a \in \overline{A}$, so $A = \overline{A}$.

On the other hand, if $A = \overline{A}$, then A is closed, since $\overline{A}$ is closed. □

Proposition 10. *Let X be a topological space and let $A \subset X$. Then A is closed if and only if every point in A is an closure point.*

Proof. This is just a rewording of the previous proposition. □

Proposition 11. *Let X be a topological space and $A \subset X$. Then $\overline{A}$ is the intersection of the closed sets of X which contain A.*

Proof. Let $\mathcal{F} = \{F \subset X \mid F \text{ is closed and } A \subset F\}$. We wish to show that $\overline{A} = \cap\mathcal{F}$.

Since $\overline{A}$ is a closed set which contains A, $\overline{A} \in \mathcal{F}$, so $\cap\mathcal{F} \subset \overline{A}$.

Now suppose $u \notin \cap\mathcal{F}$. Then $u \notin F$ for some $F \in \mathcal{F}$. Thus $u \in X \smallsetminus F$, which is open, so there exists an open neighborhood U of u which is contained in $X \smallsetminus F \subset X \smallsetminus A$. That is, U does not intersect A, and u is not a closure point of A. Thus $b \in \overline{A}$ if and only if $b \in \cap\mathcal{F}$, so $\overline{A} = \cap\mathcal{F}$. □

Proposition 12. *Let X be a topological space. Then*

(K1) $\overline{\varnothing} = \varnothing$;
(K2) $A \subset \overline{A}$;
(K3) $\overline{\overline{A}} = \overline{A}$;
(K4) $\overline{(A \cup B)} = \overline{A} \cup \overline{B}$.

Proof. The first two are immediate from the definition.

From **(K2)** we have $\overline{A} \subset \overline{\overline{A}}$. Suppose that $x \in \overline{\overline{A}}$. Then every open neighborhood of x intersects $\overline{A}$. For any open neighborhood U of x, let $y \in U \cap \overline{A}$. Then every open neighborhood of y intersects A. Since U is an open neighborhood of y, U intersects A. Thus $x \in \overline{A}$.

Suppose that $x \notin \overline{A} \cup \overline{B}$. Then there exists a neighborhoods U, V of x such that $U \cap A = \varnothing$ and $V \cap B = \varnothing$. Then $U \cap V$ is a neighborhood of x such that $(U \cap V) \cap (A \cup B) = \varnothing$. So $x \notin \overline{(A \cup B)}$. Therefore $\overline{(A \cup B)} \subset \overline{A} \cup \overline{B}$.

Suppose that $x \in \overline{A} \cup \overline{B}$. Then every open neighborhood of x intersects A or B, so it intersects $A \cup B$. Thus $x \in \overline{A} \cup \overline{B}$, so $\overline{A} \cup \overline{B} \subset \overline{(A \cup B)}$. □

Proposition 13. *Let X be a topological space. If $A \subset B \subset X$, then $\overline{A} \subset \overline{B}$.*

Proof. Let $y \in \overline{A}$. Then every neighborhood of y intersects A. Since $A \subset B$, every neighborhood of y intersects B. Thus $y \in \overline{B}$. □

Proposition 14. *Let X be a topological space and let $A \subset X$. Then*

(a) $A^\circ = X \smallsetminus \overline{(X \smallsetminus A)}$;
(b) $\overline{A} = X \smallsetminus (X \smallsetminus A)^\circ$.

Proof. Exercise. □

3.3. **Boundary Points.**

Definition 7. Let X be a topological space and let $A \subset X$. A *boundary point* of A is a point $x \in X$ such that every neighborhood of x intersects A and $X \smallsetminus A$. The *boundary* of A is the set of boundary points of A and is denoted ∂A.

Proposition 15. *Let X be a topological space and let $A \subset X$. Then*

(a) $\partial A = \overline{A} \smallsetminus A^\circ$;
(b) $\partial A = \overline{A} \cap \overline{(X \smallsetminus A)}$;
(c) $\partial A = \partial(X \smallsetminus A)$;
(d) $\overline{A} = A \cap \partial A$;
(e) $A^\circ = A \smallsetminus \partial A$;
(f) $\partial(\partial A) \subset \partial A$;
(g) $A \cap B \cap \partial(A \cap B) = A \cap B \cap (\partial A \cup \partial B)$.

Proposition 16. *Let X be a topological space and let $A \subset X$. Then $\partial A = \varnothing$ if and only if A is both open and closed.*

Proof.

($\Rightarrow$) Suppose that $\partial A = \varnothing$. Then $\overline{A} \subset A^\circ$. But $A^\circ \subset A \subset \overline{A}$, so $A^\circ = A = \overline{A}$. Thus A is both open and closed.

($\Leftarrow$) Suppose that A is both open and closed. Then $A^\circ = A = \overline{A}$, so $\partial A = \overline{A} \smallsetminus A^\circ = \varnothing$. □

3.4. **Accumulation Points.**

Definition 8. Let X be a topological space and let $A \subset X$. A *accumulation point* of A is a point $x \in X$ such that every deleted neighborhood of x intersects A. The *derived set* of A is the set of accumulation points of A and is denoted A'.

Proposition 17. *Let X be a topological space and $A, B \subset X$.*

- **(a)** $A \subset B \Rightarrow A' \subset B'$;
- **(b)** $(A \cup B)' = A' \cup B'$;
- **(c)** $\overline{A} = A \cup A'$.

Corollary 1. *A subset of a topological space is closed if and only if it contains all of its accumulation points.*

3.5. **Isolated Points.**

Definition 9. Let X be a topological space and let $A \subset X$.

An *isolated point* of A is a point $x \in A$ such that some deleted neighborhood of x is contained in $X \smallsetminus A$.

The set of isolated points of A will be denoted $\dot{A}$.

Proposition 18. *Let X be a topological space and $A \subset X$.*

- **(a)** $\dot{A} \subset A$;
- **(b)** $\dot{A} \subset \partial A$;
- **(c)** $\overline{A} = A' \sqcup \dot{A}$.

3.6. **Refinements.**

3.6.1. *Refinements.*

Definition 10. Let X be a set and let $\mathcal{S}$ and $\mathcal{T}$ be topologies on X.

If $\mathcal{S} \subset \mathcal{T}$, we say that $\mathcal{S}$ is a *courser* topology than $\mathcal{T}$ and that $\mathcal{T}$ is a *finer* topology than $\mathcal{S}$.

Remark 3. The coarsest topology on a set is the trivial topology and the finest topology on a set is the discrete topology.

Proposition 19. *Let X be a set and let $\{\mathcal{T}_\alpha \mid \alpha \in A\}$ be a collection of topologies on X. Then $\mathcal{I} = \cap_{\alpha \in A} \mathcal{T}_\alpha$ is a topology on X.*

Proof. Since the empty set and the entire set are in every topology, they are in the intersection.

If $\mathcal{U} \subset \mathcal{I}$, then $\mathcal{U} \subset \mathcal{T}_\alpha$ for every α. Thus $\cup \mathcal{U} \in \mathcal{T}_\alpha$ for every α, so $\cup \mathcal{U} \in \mathcal{I}$.

If $\mathcal{U} \subset \mathcal{I}$, then $\mathcal{U} \subset \mathcal{T}_\alpha$ for every α. If $\mathcal{U}$ is a finite collection, $\cap \mathcal{U} \in \mathcal{T}_\alpha$ for every α, so $\cap \mathcal{U} \in \mathcal{I}$. $\square$

3.6.2. *Generated Topologies.*

Definition 11. Let X be a set and let $\mathcal{A} \in \mathcal{P}(X)$.

The *topology generated by* $\mathcal{A}$ is the intersection of all the topologies on X which contain $\mathcal{A}$, and is denoted $\langle \mathcal{A} \rangle$.

Remark 4. The topology generated by a collection $\mathcal{A} \subset \mathcal{P}(X)$ is the coarsest topology on X in which all of the sets in $\mathcal{A}$ are open.

4. Continuous Functions

4.1. **Continuous Functions.**

Definition 12. Let X and Y be spaces and $f : X \to Y$.

We say that f is *continuous* if for every open set $V \subset Y$, $f^{-1}(V) \subset X$ is open.

Definition 13. Let X and Y be spaces and $f : X \to Y$ and let $x_0 \in X$.

We say that f is *continuous at* x_0 if for every neighborhood V of $f(x_0)$ there exists a neighborhood U of x_0 such that $f(U) \subset V$.

Proposition 20. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if f is continuous at every point in X.*

Proof. Suppose that f is continuous, and let $x_0 \in X$. Let V be a neighborhood of $y_0 = f(x_0)$. Then $U = f^{-1}(V)$ is a neighborhood of x_0 which maps into V.

Conversely, suppose that f is continuous at every point in X. Let $V \subset Y$ be open and let $U = f^{-1}(V)$. For every $x \in U$, V is a neighborhood of $f(x)$, so there exists an open neighborhood U_x of x such that $f(U_x) \subset V$. But then $U_x \subset U$, and U is the union of such sets; thus U is open, and f is continuous. □

Proposition 21. *Let X and Y be spaces and $f : X \to Y$. If X has the discrete topology or Y has the trivial topology then f is continuous.*

Proposition 22. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if for every closed set $F \subset Y$, $f^{-1}(F) \subset X$ is closed.*

Proof.

($\Rightarrow$) Suppose that f is continuous. Let $F \subset Y$ be closed. Let $U = Y \smallsetminus F$; then U is open, so $f^{-1}(U)$ is open, so $X \smallsetminus f^{-1}(U)$ is closed. But $X \smallsetminus f^{-1}(U) = f^{-1}(Y \smallsetminus U) = f^{-1}(F)$.

($\Leftarrow$) Suppose that for every closed set $F \subset Y$, $f^{-1}(F)$ is closed in X. Let $U \subset Y$ be open; then $Y \smallsetminus U$ is closed in Y, so $f^{-1}(Y \smallsetminus U)$ is closed in X. Thus $f^{-1}(U) = X \smallsetminus f^{-1}(Y \smallsetminus U)$ is open in X. Therefore f is continuous. □

Proposition 23. *Let X and Y be spaces and $f : X \to Y$. Then f is continuous if and only if for every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.*

Proof.

($\Rightarrow$) Suppose that f is continuous. Let $A \subset X$ and let $y \in f(\overline{A})$. Then $y = f(x)$ for some point $x \in \bar{}(A)$. Let V be an open neighborhood of y. Then $f^{-1}(V)$ is open in X and contains x. Thus there exists $a \in A \cap f^{-1}(V)$, and $f(a) \in V$; that is, V intersects $f(A)$. Therefore $y \in \overline{f(A)}$, and $f(\overline{A}) \subset \overline{f(A)}$.

($\Leftarrow$) Suppose that for every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

Let $F \subset Y$ be closed and let $A = f^{-1}(F)$. Then $f(A) = F$, and since F is closed, $\overline{f(A)} = F$. Thus $F = f(A) \subset f(\overline{A}) \subset \overline{f(A)} = F$. This shows that $f(\overline{A}) = F$, so $\overline{A} \subset f^{-1}(F) = A$; since $A \subset \overline{A}$, we see that $A = \overline{A}$, so A is closed. Therefore f is continuous. □

Proposition 24. *Let X and Y be topological spaces and let $f : X \to Y$ be continuous. Let $A \subset X$. Then*

  (a) $f(A)^\circ \subset f(A^\circ)$;
  (b) $\overset{\cdot}{f(A)} \subset f(\dot{A})$.

Proof. Let $y \in f(A)^\circ$. Then $y \in f(A)$, so $y = f(x)$ for some $x \in X$. Also there exists an open set V in Y such that $y \in V \subset f(A)$. Since f is continuous, $f^{-1}(V) \subset A$ is an open neighborhood of x contained in A, so $x \in A^\circ$, and $y \in f(A^\circ)$. Therefore $f(A)^\circ \subset f(A^\circ)$; this proves **(a)**.

Let $y \in \overset{\cdot}{f(A)})$. Then $y \in f(A)$, and there exists an open neighborhood V of y in Y such that $V \cap (f(A) \smallsetminus \{y\}) = \varnothing$. Then $f^{-1}(V) \cap A = \{y\}$, so $y \in f(\dot{A})$. Therefore $\overset{\cdot}{f(A)} \subset f(\dot{A})$; this proves **(b)**. $\square$

4.2. **Open and Closed Maps.**

Definition 14. Let X and Y be spaces and let $f : X \to Y$.

We say that f is *open* if for every open set $U \subset X$, $f(U) \subset Y$ is open.

We say that f is *closed* if for every closed set $F \subset X$, $f(F) \subset Y$ is closed.

Example 8. Let $X = \{(x, y) \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\}$ and let $Y = \mathbb{R}$. Let $f : X \to Y$ by $f(x, y) = x$. Then f is a surjective continuous closed map which is not open. It is not open because, for example, the set $\{(0, y) \mid y \in (1, 2)\}$ is open in X but projects onto a point in $\mathbb{R}$.

Example 9. Let $X = \mathbb{R}$ and $Y = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$. Define $f : X \to Y$ by $f(x) = (\cos x, \sin x)$. Then f is a surjective continuous open map which is not closed. It is open because it is open on a basis for the topology of $\mathbb{R}$ consisting of open intervals whose width is less than 2π. It is not closed because, for example, the set $\{x \in \mathbb{R} \mid x = 2\pi n + \frac{\pi}{2n}\}$ is closed in X but its image in Y has a accumulation point $(1, 0)$ which is not in the image.

Definition 15. Let X and Y be spaces and let $f : X \to Y$. We say that f is *bicontinuous* if it is both open and continuous.

Proposition 25. *Let $f : X \to Y$ be a bijective function between spaces. Then f is open if and only if f^{-1} is continuous.*

4.3. **Homeomorphisms.**

Definition 16. Let X and Y be spaces. A *homeomorphism* between X and Y is a function $f : X \to Y$ which is bijective, open, and continuous. If there exists a homeomorphism between X and Y, we say that X and Y are *homeomorphic*.

Proposition 26. *Let $(X, \mathcal{S})$ and $(Y, \mathcal{T})$ be topological spaces. Then X and Y are homeomorphic if and only if there exists an inclusion preserving bijection between $\mathcal{S}$ and $\mathcal{T}$.*

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